Generalization of the fractional Fourier transformation to an arbitrary linear lossless transformation an operator approach

1994 J. Phys. A: Math. Gen. 27 4179

(http://iopscience.iop.org/0305-4470/27/12/023)

J. Phys. A: Math. Gen. 27 (1994) 4179–4187. Printed in the UK

Generalization of the fractional Fourier transformation to an arbitrary linear lossless transformation: an operator approach

Sumiyoshi Abe† and John T Sheridan‡

† College of Science and Technology, Nihon University, 7-24-1 Narashinodai, Funabashi, Chiba 274, Japan

‡ Angewandte Optik, Physikalisches Institut, Universität Erlangen-Nürnberg, Staudstrasse 7, D-91058 Erlangen, Germany

Received 13 September 1993, in final form 1 March 1994

Abstract. We develop the operator formalism to show how systematically the fractional Fourier transformation of a wavefunction, recently introduced by Namias, can be derived from the rotation of the corresponding Wigner distribution function in phase space. In this formalism, the phase factor obtained by McBride and Kerr is seen to come from the caustics of the harmonic oscillator Green function. Then the idea is generalized to the case of an arbitrary area-preserving linear transformation in phase space, and a concept of the special affine Fourier transformation (SAFT) is introduced. An explicit form of the integral representation of the SAFT is given, and some simple examples are presented.

1. Introduction

The Fourier transformation provides a powerful tool with which various wave phenomena can be analysed. One of its important properties is that it transforms a function u to itself if applied four times:

$$[FFFFu](x) = [F^4u](x) = u(x). \tag{1}$$

Recently Namias [1] has introduced a new concept of the fractional Fourier transformation (abbreviated here as FRT and symbolized by F_θ). In this theory, equation (1) is generalized to a fractional index α

$$[(F_\theta)^\alpha u](x) = u(x) \tag{2}$$

with the semigroup property

$$F_{\theta_1} F_{\theta_2} = F_{\theta_1+\theta_2} \tag{3}$$

where θs are arbitrary real numbers. An explicit integral representation of this transformation has been given [1], and its rigorous mathematical formulation has been also performed [2].

Quite recently, it has been pointed out [3] that there is a close relationship between the FRT of a function u and the rotation of the corresponding Wigner distribution function (WDF) in phase space by the angle fractional of $\pi/2$.

0305-4470/94/124179+09$19.50

In this paper, we investigate the FRT based on the operator formalism and discuss a further generalization of Namias' idea. The main advantages of the present approach are (a) it enables us to see the above-mentioned relationship between the FRT and the rotation of the WDF in a transparent geometric manner, (b) the phase factor, which has been introduced by McBride and Kerr [2] in order to make Namias' original definition of the FRT rigorous, can be interpreted naturally as the effect of the caustics on the harmonic oscillator Green function, (c) then it is quite straightforward to include a more general transformation (the special affine transformation) in phase space, in which the WDF is defined. The corresponding transform of a given function will be referred to in this paper as the *special affine Fourier transformation* (SAFT). We derive the explicit form of its integral representation. We use the quantum mechanical notation [4] for convenience. However, our discussion itself is, of course, not restricted to quantum theory but applicable to any wave phenomena.

The paper is organized as follows. In section 2, first the basics of the quantum mechanical notation relevant to our discussion are summarized. Then a brief review is given for the construction of the WDF. In section 3, the relation between the FRT and the rotation of the WDF in phase space is shown in the abstract operator formalism. Section 4, which is the main part of the paper, contains the generalization of the FRT to the SAFT and the construction of the explicit form of the corresponding integral representation. Some simple examples are also given there. Section 5 is devoted to concluding remarks.

2. The Wigner distribution function

In the quantum mechanical notation [4], the Fourier transformation of a function $u(x) = \langle x|u\rangle$ is nothing but a change of representaiton of a given abstract state vector $|u\rangle$:

$$[Fu](k) \equiv u^F(k) = \langle k|u\rangle = \frac{1}{\sqrt{2\pi}} \int \mathrm{d}x\, \langle k|x\rangle\langle x|u\rangle. \tag{4}$$

Here $|x\rangle$ and $|k\rangle$ are base elements of the position and wavenumber representations, respectively. They are the eigenstates associated with the eigenvalues x and k of the Hermitian position and wavenumber operators, $\hat{x}$ and $\hat{k}$, satisfying the cummutation relation

$$[\hat{x}, \hat{k}] = \hat{x}\hat{k} - \hat{k}\hat{x} = \mathrm{i}. \tag{5}$$

(In quantum theory, $\hat{k}$ is the momentum operator divided by $\hbar$.) Some basic formulae are presented as follows:

$$\langle x|x'\rangle = \delta(x - x') \tag{6}$$

$$\langle k|k'\rangle = \delta(k - k') \tag{7}$$

$$\langle x|k\rangle = \langle k|x\rangle^* = \frac{1}{\sqrt{2\pi}} \exp(\mathrm{i}kx) \tag{8}$$

$$\int \mathrm{d}x\, |x\rangle\langle x| = 1 \tag{9}$$

$$\int \mathrm{d}k\, |k\rangle\langle k| = 1. \tag{10}$$

The WDF [5, 6] associated with the state $|u\rangle$ is defined by

$$W(x,k)=\frac{1}{2\pi}\int \mathrm{d}y\,\exp(-\mathrm{i}ky)u^*\left(x-\frac{y}{2}\right)u\left(x+\frac{y}{2}\right). \tag{11}$$

Let us see that this function has also the following operator form:

$$W(x,k)=\mathrm{Tr}[\hat{\rho}\hat{\Delta}(x,k)] \tag{12}$$

where $\hat{\rho}$ is the density operator

$$\hat{\rho}=|u\rangle\langle u| \tag{13}$$

and $\hat{\Delta}$ is the Wigner operator

$$\hat{\Delta}(x,k)=\frac{1}{(2\pi)^2}\iint \mathrm{d}s\,\mathrm{d}t\,\exp[\mathrm{i}s(\hat{x}-x)+\mathrm{i}t(\hat{k}-k)]. \tag{14}$$

(Equation (13) can be extended to the case of incoherent waves where $\hat{\rho}$ is not factorizable.) We express the trace operation in (12) in the position representation and insert partition of unity (9) between $\hat{\rho}$ and $\hat{\Delta}$ to obtain

$$W(x,k)=\int\int \mathrm{d}x'\,\mathrm{d}x''\,u^*(x')u(x'')\omega(x',x'':x,k) \tag{15}$$

$$\omega(x',x'':x,k)=\frac{1}{(2\pi)^2}\iint \mathrm{d}s\,\mathrm{d}t\,\exp[-\mathrm{i}(sx+tk)]\langle x'|\exp[\mathrm{i}(s\hat{x}+t\hat{k})]|x''\rangle. \tag{16}$$

From the formula

$$\exp(\hat{B}+\hat{C})=(\exp\hat{B})\exp\left(\hat{C}-\frac{1}{2!}[\hat{B},\hat{C}]+\frac{1}{3!}[\hat{B},[\hat{B},\hat{C}]]-\cdots\right) \tag{17}$$

it follows that

$$\begin{aligned}\langle x'|\exp[\mathrm{i}(s\hat{x}+t\hat{k})]|x''\rangle&=\exp\left(\frac{\mathrm{i}}{2}st\right)\langle x'|\exp(\mathrm{i}s\hat{x})\exp(\mathrm{i}t\hat{k})|x''\rangle\\&=\exp\left(\frac{\mathrm{i}}{2}st+\mathrm{i}sx'\right)\delta(t+x'-x'')\end{aligned} \tag{18}$$

where the formula $\langle x'|\exp(\mathrm{i}t\hat{k})=\langle x'+t|$ has been used. Therefore we have

$$\omega(x',x'':x,k)=\frac{1}{2\pi}\exp[\mathrm{i}k(x'-x'')]\delta\left(x-\frac{x'+x''}{2}\right). \tag{19}$$

Now with a simple calculation, one can see that (15) combined with (19) leads to definition (11).

It is known that the Wigner operator defines the following operator-correspondence relations:

$$\hat{x}\hat{\Delta}(x,k)=\left(x-\frac{\mathrm{i}}{2}\frac{\partial}{\partial k}\right)\hat{\Delta}(x,k)\qquad \hat{\Delta}(x,k)\hat{x}=\left(x+\frac{\mathrm{i}}{2}\frac{\partial}{\partial k}\right)\hat{\Delta}(x,k) \tag{20}$$

$$\hat{k}\hat{\Delta}(x,k)=\left(k+\frac{\mathrm{i}}{2}\frac{\partial}{\partial x}\right)\hat{\Delta}(x,k)\qquad \hat{\Delta}(x,k)\hat{k}=\left(k-\frac{\mathrm{i}}{2}\frac{\partial}{\partial x}\right)\hat{\Delta}(x,k). \tag{21}$$

These formulae can be used to relate the geometric transformations in phase space to the unitary transformations in the Hilbert space, and vice versa.

3. An operator approach to the fractional Fourier transformation

Let us consider the transformation of the WDF under the rotation of an angle θ in phase space (x, k), that is,

$$W(x,k) \to W^R(x,k) = W(x\cos\theta - k\sin\theta, k\cos\theta + x\sin\theta) \tag{22}$$

which, for an infinitesimal rotation angle $\delta\theta$, becomes

$$W(x,k) \to W^{IR}(x,k) = \left[1 + \delta\theta\left(x\frac{\partial}{\partial k} - k\frac{\partial}{\partial x}\right)\right] W(x,k). \tag{23}$$

From (12), (20), and (21), the right-hand side of this equation is rewritten as

$$\begin{aligned} W^{IR}(x,k) &= \mathrm{Tr}\left\{\hat{\rho}\left[1 + \delta\theta\left(x\frac{\partial}{\partial k} - k\frac{\partial}{\partial x}\right)\right]\hat{\Delta}(x,k)\right\} \\ &= \mathrm{Tr}\left\{\hat{\rho}\left[1 + \frac{i}{2}\delta\theta(\hat{k}^2 + \hat{x}^2)\right]\hat{\Delta}(x,k)\left[1 - \frac{i}{2}\delta\theta(\hat{k}^2 + \hat{x}^2)\right]\right\} \\ &= \mathrm{Tr}\left\{\left[1 - \frac{\mathrm{i}}{2}\delta\theta(\hat{k}^2 + \hat{x}^2)\right]\hat{\rho}\left[1 + \frac{\mathrm{i}}{2}\delta\theta(\hat{k}^2 + \hat{x}^2)\right]\hat{\Delta}(x,k)\right\} \end{aligned} \tag{24}$$

where the cyclicity property of the trace operation has been used in the last equality. The finite rotation (22) is an accumulation of infinitesimal rotations, and is expressed as follows:

$$W(x,k) \to W^R(x,k) = \mathrm{Tr}[\hat{U}(\theta)\hat{\rho}\hat{U}^{+}(\theta)\hat{\Delta}(x,k)] \tag{25}$$

$$\hat{U}(\theta) = \exp\left[-\frac{\mathrm{i}}{2}\theta(\hat{k}^2 + \hat{x}^2)\right]. \tag{26}$$

Therefore the rotation of the WDF corresponds to a unitary transformation of $|u\rangle$

$$|u\rangle \to |u^R\rangle = \exp(\mathrm{i}\chi)\hat{U}(\theta)|u\rangle. \tag{27}$$

Here the overall constant phase factor has been introduced, since equation (25) does not know anything about it. (This phase ambiguity will be eliminated by semigroup property (3). See below.) Using (9), we have the following x-representation for the rotated wave:

$$u^R(x) = \langle x|u^R\rangle = \mathrm{e}^{\mathrm{i}\chi}\int \mathrm{d}x'\, G(x,x';\theta)u(x') \tag{28}$$

where

$$G(x,x';\theta) = \langle x|\hat{U}(\theta)|x'\rangle \tag{29}$$

with the 'initial' condition $G(x,x';0) = \delta(x-x')$. Apart from the multi-valuedness of θ, identifying θ with time, we see that the kernel $G(x,x';\theta)$ is formally equal to the Green function of the harmonic oscillator Schrödinger equation with unit mass and frequency [7]:

$$G(x,x';\theta) = \frac{1}{\sqrt{2\pi|\sin\theta|}}\exp\left\{-\mathrm{i}\frac{\pi}{2}\left[\frac{1}{2} + I\left(\frac{\theta}{\pi}\right)\right] + \frac{\mathrm{i}}{2\sin\theta}[(x^2 + x'^2)\cos\theta - 2xx']\right\}. \tag{30}$$

In the above, the symbol $I(A)$ stands for the largest integer smaller than A. This factor is known to come from the caustics, and plays a central role in the proof of the (semi)group property

$$G(x_1, x_2; \theta_1 + \theta_2) = \int dx_3\, G(x_1, x_3; \theta_1) G(x_3, x_2; \theta_2). \tag{31}$$

Clearly this has its origin from the Abelian-group property of the unitary operator $\hat{U}$

$$\hat{U}(\theta_1)\hat{U}(\theta_2) = \hat{U}(\theta_1 + \theta_2). \tag{32}$$

These mean that the (semi)group property of the transformation (27) (or (28)) is established without any extra phase factor. Therefore, $\exp(i\chi) = 1$ for the FRT.

Thus we have

$$u^R(x) = \frac{1}{\sqrt{2\pi|\sin\theta|}} \exp\left\{-i\frac{\pi}{2}\left[\frac{1}{2} + I\left(\frac{\theta}{\pi}\right)\right] + \frac{i}{2}x^2\cot\theta\right\} \times \int dx' \exp\left(\frac{i}{2}x'^2\cot\theta - ixx'\operatorname{cosec}\theta\right) u(x'). \tag{33}$$

This is essentially equivalent to the definition of the FRT of u with $\alpha = 2\pi/\theta$.†

In this construction, the constant phase factor in the FRT, which has been introduced by McBride and Kerr [2] to make Namias' original formulation rigorous, is seen to come from the caustics of the harmonic oscillator Green function.

In the special case of $\theta = \pi/2$, equation (33) is reduced to the classical Fourier transformation.

4. The special affine Fourier transformation

The present operator approach to the FRT enables us to construct a more general class of transformations. In this section, we consider a special affine transformation in phase space and introduce a concept of the special affine Fourier transformation (SAFT).

A special affine transformation in phase space (x, k) is given by

$$\begin{pmatrix} x \\ k \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ k \end{pmatrix} + \begin{pmatrix} m \\ n \end{pmatrix} \tag{34}$$

with the 'lossless' (area-preserving, or power-preserving) condition

$$ad - bc = 1. \tag{35}$$

The set of such transformations forms the inhomogeneous special linear group ISL(2, $\mathbf{R}$). Under this transformation, the Wigner operator changes as follows:

$$\begin{aligned}\hat{\Delta}(x,k) &\to \frac{1}{(2\pi)^2}\int\int ds\, dt\, \exp\{is[\hat{x} - (ax + bk + m)] + it[\hat{k} - (cx + dk + n)]\} \\ &= \frac{1}{(2\pi)^2}\int\int ds\, dt\, \exp\{is[d(\hat{x} - m) - b(\hat{k} - n) - x] + it[a(\hat{k} - n) - c(\hat{x} - m) - k]\} \\ &= \hat{D}^+(m,n)\frac{1}{(2\pi)^2}\int\int ds\, dt\, \exp[is(d\hat{x} - b\hat{k} - x) + it(a\hat{k} - c\hat{x} - k)]\hat{D}(m,n)\end{aligned} \tag{36}$$

† The operator in the exponential of (26) is the Hamiltonian operator of a harmonic oscillator with unit mass and frequency. This clarifies why the function $\exp(-x^2/2)H_n(x)$ (H_n: the Hermite polynomial of order n) is an eigenfunction of the FRT [1, 2].

where $\hat{D}(m,n)$ is the unitary displacement operator

$$\hat{D}(m,n) = \exp[\mathrm{i}(m\hat{k} - n\hat{x})] \tag{37}$$

provided that, in the last equality in (36), the following formula has been used:

$$(\exp\hat{B})\hat{C}\exp(-\hat{B}) = \hat{C} + [\hat{B},\hat{C}] + \frac{1}{2!}[\hat{B},[\hat{B},\hat{C}]] + \frac{1}{3!}[\hat{B},[\hat{B},[\hat{B},\hat{C}]]] + \cdots. \tag{38}$$

Next we introduce the operator

$$\hat{V}(\alpha,\beta,\gamma) = \exp(-\mathrm{i}\hat{A}) \tag{39}$$

$$\hat{A} = \frac{1}{2}[\alpha\hat{k}^2 + \beta(\hat{k}\hat{x} + \hat{x}\hat{k}) + \gamma\hat{x}^2]. \tag{40}$$

Using (38), we have

$$\hat{V}^{+}\hat{x}\hat{V} = \begin{cases} [\cos\Phi + (\beta/\Phi)\sin\Phi]\cdot\hat{x} + (\alpha/\Phi)\sin\Phi\cdot\hat{k} & (\alpha\gamma - \beta^2 > 0) \\ (1+\beta)\hat{x} + \alpha\hat{k} & (\alpha\gamma - \beta^2 = 0) \\ [\cosh\Phi + (\beta/\Phi)\sinh\Phi]\cdot\hat{x} + (\alpha/\Phi)\sinh\Phi\cdot\hat{k} & (\alpha\gamma - \beta^2 < 0) \end{cases} \tag{41}$$

$$\hat{V}^{+}\hat{k}\hat{V} = \begin{cases} [\cos\Phi - (\beta/\Phi)\sin\Phi]\cdot\hat{k} - (\gamma/\Phi)\sin\Phi\cdot\hat{x} & (\alpha\gamma - \beta^2 > 0) \\ (1-\beta)\hat{k} - \gamma\hat{x} & (\alpha\gamma - \beta^2 = 0) \\ [\cosh\Phi - (\beta/\Phi)\sinh\Phi]\cdot\hat{k} - (\gamma/\Phi)\sinh\Phi\cdot\hat{x} & (\alpha\gamma - \beta^2 < 0) \end{cases} \tag{42}$$

$$\Phi \equiv \sqrt{|\alpha\gamma - \beta^2|}. \tag{43}$$

The case $\alpha\gamma - \beta^2 = 0$ can be seen as the limit $\Phi \to 0$ in (41) and (42). Henceforth, the cases $\alpha\gamma - \beta^2 \neq 0$ are considered.

Now, with the identifications of coefficients

$$\begin{cases} a = \cos\Phi - (\beta/\Phi)\sin\Phi & b = -(\alpha/\Phi)\sin\Phi \\ c = (\gamma/\Phi)\sin\Phi & d = \cos\Phi + (\beta/\Phi)\sin\Phi \end{cases} \qquad (\alpha\gamma - \beta^2 > 0) \tag{44}$$

$$\begin{cases} a = \cosh\Phi - (\beta/\Phi)\sinh\Phi & b = -(\alpha/\Phi)\sinh\Phi \\ c = (\gamma/\Phi)\sinh\Phi & d = \cosh\Phi + (\beta/\Phi)\sinh\Phi \end{cases} \qquad (\alpha\gamma - \beta^2 < 0) \tag{45}$$

equation (36) is further rewritten as

$$\hat{\Delta}(x,k) \to \hat{D}^{+}(m,n)\hat{V}^{+}(\alpha,\beta,\gamma)\hat{\Delta}(x,k)\hat{V}(\alpha,\beta,\gamma)\hat{D}(m,n). \tag{46}$$

Therefore, repeating the discussions in section 3 about the transformations of the WDF and its corresponding wavefunction, we obtain

$$u^{\mathrm{SAFT}}(x) = \exp\left(\mathrm{i}\chi + \frac{\mathrm{i}}{2}mn\right)\int \mathrm{d}x'\,\exp(-\mathrm{i}nx')\langle x|\hat{V}(\alpha,\beta,\gamma)|x'-m\rangle u(x') \tag{47}$$

where the formula $\hat{D}(m,n)|x'\rangle = \exp(-\mathrm{i}nx' + \mathrm{i}mn/2)|x'-m\rangle$ has been used, and the arbitrary phase factor, $\exp(\mathrm{i}\chi)$, has been introduced again.

To calculate the kernel in (47), it is convenient to diagonalize the quadratic form (40) with respect to $\hat{x}$ and $\hat{k}$. As can be seen, it is diagonalized by a rotation operator of the form (26).

In the case when $\beta > 0$,

$$\hat{U}^{+}(\phi)\hat{A}\hat{U}(\phi) = \tfrac{1}{2}(\lambda_{+}\hat{k}^{2} + \lambda_{-}\hat{x}^{2}) \tag{48}$$

$$\lambda_{\pm} = \tfrac{1}{2}[\alpha + \gamma \pm \sqrt{(\alpha - \gamma)^{2} + 4\beta^{2}} \tag{49}$$

$$\cos\phi = \sqrt{\frac{\alpha - \lambda_{-}}{\lambda_{+} - \lambda_{-}}} \qquad \sin\phi = \sqrt{\frac{\lambda_{+} - \alpha}{\lambda_{+} - \lambda_{-}}}. \tag{50}$$

In terms of $\lambda_{\pm}$, Φ in (43) is expressed as

$$\Phi = \sqrt{|\lambda_{+}\lambda_{-}|}. \tag{51}$$

On the other hand, when $\beta < 0$, we obtain $\hat{U}(\phi)\hat{A}\hat{U}^{+}(\phi) = (1/2)(\lambda_{+}\hat{k}^{2} + \lambda_{-}\hat{x}^{2})$ with the same $\lambda_{\pm}$ and ϕ. Because of the property $\hat{U}(-\phi) = \hat{U}^{+}(\phi)$, the $\beta < 0$ case is obtained by replacing ϕ by $-\phi$ in the case $\beta > 0$. So, it is sufficient to discuss only the case $\beta > 0$.

Using (48), the kernel in integral (47) can be written as follows:

$$\langle x|\hat{V}(\alpha,\beta,\gamma)|x' - m\rangle = \iint \mathrm{d}x_1\,\mathrm{d}x_2\, G(x, x_1; \phi) G^{*}(x' - m, x_2; \phi) K(x_1, x_2) \tag{52}$$

$$K(x_1, x_2) = \langle x_1|\exp\left[-\frac{\mathrm{i}}{2}(\lambda_{+}\hat{k}^{2} + \lambda_{-}\hat{x}^{2})\right]|x_2\rangle \tag{53}$$

where G is given in (29) with (26).

The function K is classified into two types. If both λ_{+} and λ_{-} are positive (negative), then $K(K^{*})$ is essentially the Green function of a normal harmonic oscillator. If $\lambda_{+} > 0 > \lambda_{-}$ ($\lambda_{-} > 0 > \lambda_{+}$), then $K(K^{*})$ is the Green function of an upside-down oscillator [8]. That is, for the positive $\lambda_{\pm}$

$$\begin{aligned} K(x_1, x_2) = {} & \exp\left\{-\mathrm{i}\frac{\pi}{2}\left[\frac{1}{2} + I\left(\frac{\Phi}{\pi}\right)\right]\right\}\sqrt{\frac{R}{2\pi|\sin\Phi|}} \\ & \times \exp\left\{\frac{\mathrm{i}R}{2\sin\Phi}[(x_1^{2} + x_2^{2})\cos\Phi - 2x_1x_2]\right\} \end{aligned} \tag{54}$$

with

$$R \equiv \sqrt{|\lambda_{-}/\lambda_{+}|} = \Phi/|\lambda_{+}| = |\lambda_{-}|/\Phi \tag{55}$$

while, for $\lambda_{+} > 0 > \lambda_{-}$

$$K(x_1, x_2) = \exp\left(-\mathrm{i}\frac{\pi}{4}\right)\sqrt{\frac{R}{2\pi\sinh\Phi}}\exp\left\{\frac{\mathrm{i}R}{2\sinh\Phi}[(x_1^{2} + x_2^{2})\cosh\Phi - 2x_1x_2]\right\} \tag{56}$$

with R given in (55).

Substituting the above expressions for K into (52) and using the integral formulae

$$\int_{-\infty}^{\infty} \mathrm{d}x\ \exp[\mathrm{i}(rx^2+sx)] = \begin{cases} \sqrt{\dfrac{\pi}{r}} \exp\left(\dfrac{\mathrm{i}}{4}\pi - \dfrac{\mathrm{i}s^2}{4r}\right) & (r>0) \\ \sqrt{\dfrac{\pi}{-r}} \exp\left(-\dfrac{\mathrm{i}}{4}\pi - \dfrac{\mathrm{i}s^2}{4r}\right) & (r<0) \end{cases} \qquad (57)$$

we have, for $\lambda_+ > \lambda_- > 0$ ($\beta > 0$)

$$\begin{aligned} \langle x|\hat{V}(\alpha,\beta,\gamma)|x'-m\rangle &= \frac{1}{\sqrt{2\pi|\sin\Phi(R\sin^2\phi + R^{-1}\cos^2\phi)|}} \exp\left\{-\mathrm{i}\frac{\pi}{2}\left[\frac{1}{2} + I\left(\frac{\Phi}{\pi}\right)\right]\right\} \\ &\quad \times \exp\left[\frac{\mathrm{i}}{2\sin\Phi(R\sin^2\phi + R^{-1}\cos^2\phi)}\{[x^2+(x'-m)^2]\cos\Phi - 2x(x'-m) \right. \\ &\quad \left. + [x^2-(x'-m)^2](R-R^{-1})\sin\Phi\sin\phi\cos\phi\}\right] \end{aligned} \qquad (58)$$

while, for $\lambda_+ > 0 > \lambda_-$ ($\beta > 0$),

$$\begin{aligned} \langle x|\hat{V}(\alpha,\beta,\gamma)|x'-m\rangle &= \frac{1}{\sqrt{2\pi\sinh\Phi|R^{-1}\cos^2\phi - R\sin^2\phi|}} \exp\left(-\mathrm{i}\frac{\pi}{4} + \mathrm{i}\delta\right) \\ &\quad \times \exp\left[\frac{\mathrm{i}}{2\sinh\Phi(R^{-1}\cos^2\phi - R\sin^2\phi)}\{[x^2+(x'-m)^2]\cosh\Phi - 2x(x'-m) \right. \\ &\quad \left. + [(x'-m)^2 - x^2](R+R^{-1})\sinh\Phi\sin\phi\cos\phi\}\right] \end{aligned} \qquad (59)$$

with $\delta = 0$ ($\alpha > 0$, i.e. $b < 0$), $3\pi/4$ ($\alpha < 0$, i.e. $b > 0$, and $R\coth\Phi - \cot\phi > 0$), $-\pi/2$ ($\alpha < 0$, i.e. $b > 0$, and $R\coth\Phi - \cot\phi < 0$).

Here we recall that there is the ambiguity concerning the overall constant phase in (47). In the case of the FRT in section 3, the phase factor was determined in conformity with (semi)group property (31) or (32). However, in the present discussion of the SAFT, such a property does not exist because of the non-Abelian nature of ISL(2, **R**). Therefore, in what follows, the constant phase factors in (58) and (59) are absorbed into the factor $\exp(\mathrm{i}\chi)$ in (47), which is further set equal to unity for simplicity.

Now let us express the kernel $\langle x|\hat{V}|x'-m\rangle$ in terms of the parameters of transformation (34). From (44), (45), (50), and (55), we find that, up to the irrelevant constant phase factor, both (58) and (59) have the following form:

$$\langle x|\hat{V}|x'-m\rangle = \frac{1}{\sqrt{2\pi|b|}} \exp\left\{-\frac{\mathrm{i}}{2b}[ax^2 + d(x'-m)^2 - 2x(x'-m)]\right\} \qquad (60)$$

which turns out to hold for any values of ISL(2, **R**) parameters.

Thus we obtain the main result:

$$\begin{aligned} u^{\mathrm{SAFT}}(x) &= \frac{1}{\sqrt{2\pi|b|}} \exp\left[-\frac{\mathrm{i}}{b}\left(\frac{a}{2}x^2 + mx\right)\right] \\ &\quad \times \int \mathrm{d}x'\ \exp\left\{-\frac{\mathrm{i}}{2b}[\mathrm{d}x'^2 - 2(x+dm-bn)x']\right\} u(x'). \end{aligned} \qquad (61)$$

In the special case of $b = -c = -\sin\theta$, $a = d = \cos\theta$, and $m = n = 0$, equation (61) reduces to the main part of the FRT (33) without the constant phase factor.

Finally we present two simple examples of the SAFT:

[I] $u(x) = \exp(\mathrm{i}px)$

$$u^{\mathrm{SAFT}}(x) = \frac{1}{\sqrt{|d|}} \exp\left\{-\frac{\mathrm{i}}{2bd}[(ad-1)x^2 - 2b(p-n)x]\right\} \tag{62}$$

[II] $u(x) = \exp(-\frac{1}{2}\sigma x^2) \qquad (\sigma > 0)$

$$u^{\mathrm{SAFT}}(x) = (b^2\sigma^2 + d^2)^{-1/4} \exp\left[-\frac{\sigma}{2}\frac{(dm-bn)^2}{b^2\sigma^2+d^2}\right] \times \exp\left\{-\frac{1}{2b(b\sigma+\mathrm{i}d)}[(1-ad+\mathrm{i}ab\sigma)x^2 - 2b(n-\mathrm{i}m\sigma)x]\right\} \tag{63}$$

provided that all constant phase factors have been neglected.

5. Concluding remarks

We have developed the operator formalism to show that the fractional Fourier transformation introduced by Namias is obtained from the rotation of the Wigner distribution function in phase space. We have seen that the phase factor in the FRT obtained by McBride and Kerr can be interpreted as the effect of the caustics for the harmonic oscillator Green function. We have generalized further the FRT to the special affine Fourier transformation, in which a transformed wavefunction corresponds to the ISL(2, **R**) transformation of the WDF in phase space, and have given an explicit form of its integral representation.

Recently it has been shown [3, 9] that an optical device with graded-index media can perform the FRT of light waves. The SAFT presented here is the most general lossless (inhomogeneous) linear transformation in phase space. We expect that the SAFT can provide fresh geometric insight in the areas of signal processing and beam shaping.

Acknowledgments

One of the authors (SA) thanks the Futaba Electronics Memorial Foundation for financial support. JTS would like to thank the European Commission for the provision of a subsidy. This work was initiated while we held Humboldt fellowships at Erlangen. We would like to express our sincere thanks to the Alexander von Humboldt Foundation for having given us the opportunity for this collaboration.

References

[1] Namias V 1980 *J. Inst. Math. Appl.* **25** 241
[2] McBride A C and Kerr F H 1987 *IMA J. Appl. Math.* **39** 159
[3] Lohmann A W 1993 *J. Opt. Soc. Am.* A **10** 2181
[4] Dirac P A M 1958 *The Principles of Quantum Mechanics* (Oxford: Clarendon)
[5] Wigner E 1932 *Phys. Rev.* **40** 749
[6] Balazs N L and Jennings B K 1984 *Phys. Rep.* **104** 347
[7] Dittrich W and Reuter M 1992 *Classical and Quantum Dynamics* (Heidelberg: Springer)
[8] Barton G 1986 *Ann. Phys.* **166** 322
[9] Mendlovic D and Ozaktas H M 1993 *J. Opt. Soc. Am.* A **10** 1875, 2522